

Rotation Theorem in Complex Numbers



Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

To rotate a point z about any centre z_0 : shift, turn, and shift back. In one line, $z' = z_0 + (z - z_0)e^{i\theta}$. On a triangle, the same idea gives the rotation theorem, which finds the third vertex of an equilateral triangle and the missing vertices of a square.

CENTRE

The fixed point we rotate about. It does not move.

Every other point keeps its distance from the centre while it turns.

THE ARROW $z - z_0$

The arrow from z_0 to z is the complex number $z - z_0$.

Start at z_0 , add $z - z_0$, and you reach z .

The arrow from z_0 to z

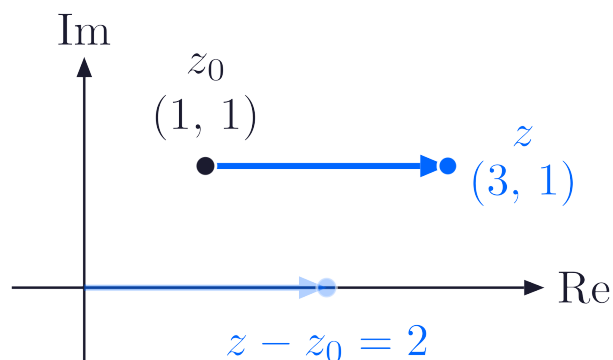
Take the centre $z_0 = 1 + i$, the point $(1, 1)$, and the point $z = 3 + i$. The arrow from z_0 to z is

$$z - z_0 = (3 + i) - (1 + i) = 2.$$

Slide this arrow so that it starts at the origin: it ends at 2. So $z - z_0$ is the same arrow, only moved to the origin.

Turning an arrow: multiply by $e^{i\theta}$

$e^{i\theta} = \cos \theta + i \sin \theta$ is the arrow of length 1 at angle θ . When two complex numbers are multiplied, the lengths multiply and the angles add, so multiplying by $e^{i\theta}$ turns an arrow by θ and keeps its length. With $\theta = \frac{\pi}{2}$, $e^{i\frac{\pi}{2}} = i$: multiplying by i is a quarter turn, and the arrow 2 turns to $2i$.



Why multiplying by i is not enough

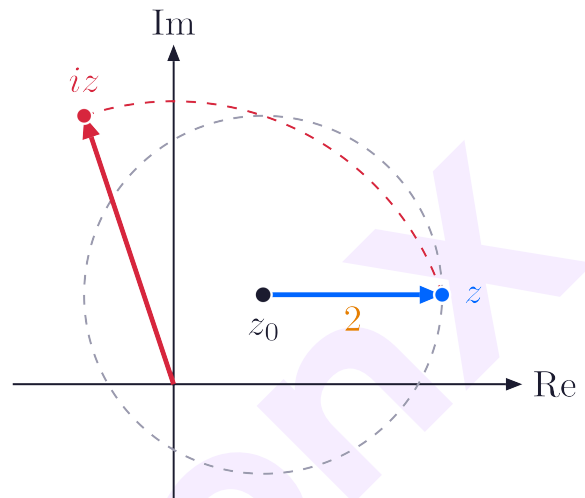
Rotate $z = 3 + i$ about the centre $z_0 = 1 + i$ by $\frac{\pi}{2}$ anticlockwise. Rotating about z_0 never changes the distance from z_0 , so the answer must lie on the circle of radius 2 about z_0 .

The common mistake

$$iz = i(3 + i) = 3i + i^2 = -1 + 3i.$$

But multiplying by i turns the arrow that starts at the origin. So the point turned about the origin, not about z_0 , and it lands off the circle:

$$|(-1 + 3i) - (1 + i)| = |-2 + 2i| = 2\sqrt{2}, \text{ not } 2.$$

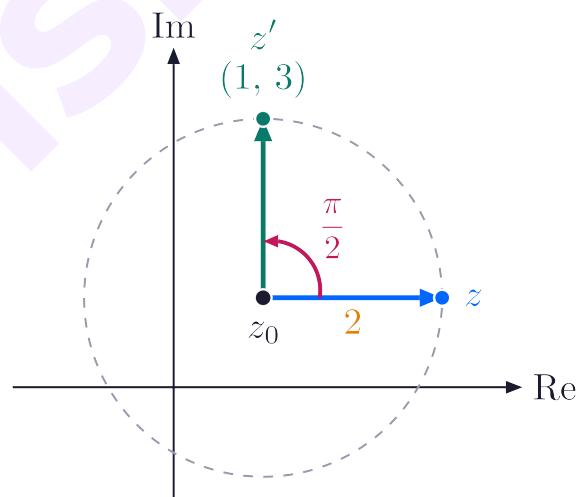


Rotation about any point: shift, turn, shift back

Shift. Slide the arrow so that the centre z_0 sits on the origin. The arrow is now the number $z - z_0 = 2$.

Turn. The arrow starts at the origin, so multiplying by i turns it correctly: $2 \times i = 2i$.

Shift back. Add z_0 again: $z' = 2i + (1 + i) = 1 + 3i$, the point $(1, 3)$. It lies on the circle, and the turn about z_0 is exactly $\frac{\pi}{2}$.



Rotation about any centre

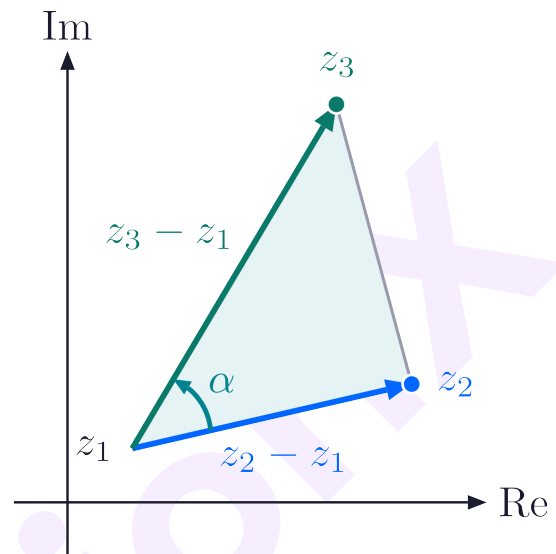
To rotate z about z_0 through an angle θ :

$$\text{anticlockwise: } z' = z_0 + (z - z_0) e^{i\theta}, \quad \text{clockwise: } z' = z_0 + (z - z_0) e^{-i\theta}.$$

To also multiply the distance from z_0 by $k > 0$, multiply by $k e^{i\theta}$ instead. A half turn ($\theta = \pi$, $e^{i\pi} = -1$) sends z to $2z_0 - z$.

The rotation theorem

Take any triangle with vertices z_1 , z_2 and z_3 , and let α be its angle at z_1 . The side from z_1 to z_2 is the arrow $z_2 - z_1$, and the side from z_1 to z_3 is the arrow $z_3 - z_1$. Turn the first arrow about z_1 anticlockwise by α : now it points along the second side. Then stretch it by the ratio of the two lengths, $\frac{|z_3 - z_1|}{|z_2 - z_1|}$, and it reaches z_3 .



Rotation theorem

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$$

The ratio of the lengths is the stretch and $e^{i\alpha}$ is the turn. Some books call it the rotation formula.

Which way is the angle measured?

Always go from the side in the denominator to the side in the numerator. Anticlockwise is $+\alpha$ and clockwise is $-\alpha$. With the sides swapped, the turn from $z_3 - z_1$ back to $z_2 - z_1$ is clockwise:

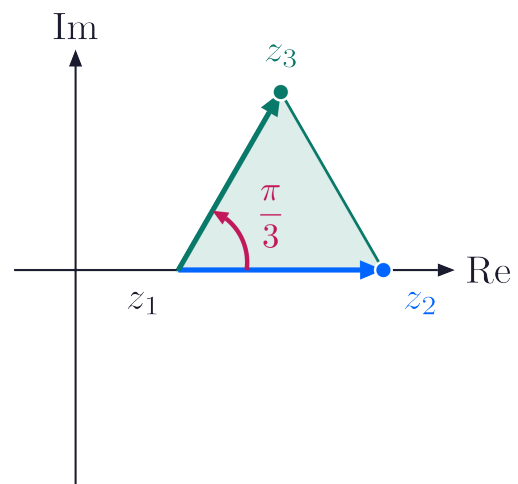
$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{|z_2 - z_1|}{|z_3 - z_1|} e^{-i\alpha}.$$

The third vertex of an equilateral triangle

Two vertices are $z_1 = 1$ and $z_2 = 3$. All three sides are equal, so there is no stretch. Every angle is $\frac{\pi}{3}$, so turn the side from z_1 to z_2 about z_1 anticlockwise by $\frac{\pi}{3}$:

$$z_3 = z_1 + (z_2 - z_1) e^{i\frac{\pi}{3}} = 1 + 2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) = 2 + \sqrt{3} i.$$

Check: $|z_3 - 1| = |1 + \sqrt{3} i| = 2$ and $|z_3 - 3| = |-1 + \sqrt{3} i| = 2$. All three sides are 2.



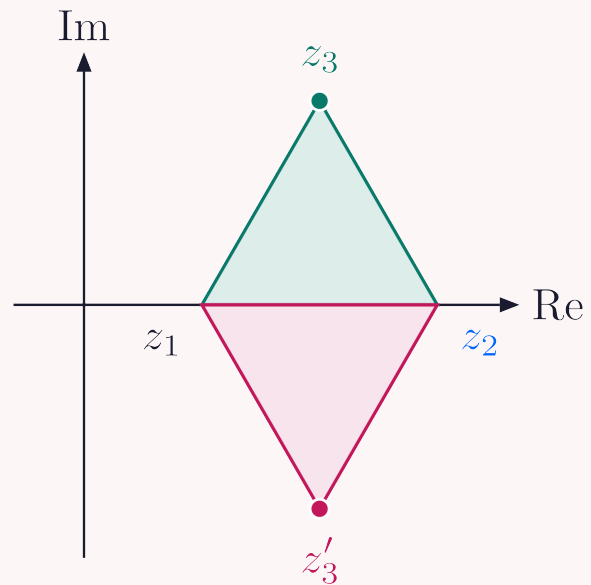
CAREFUL

One side, two triangles. Most students stop at $z_3 = 2 + \sqrt{3}i$. But the side can also turn clockwise, by $-\frac{\pi}{3}$:

$$z'_3 = 1 + 2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) = 2 - \sqrt{3}i.$$

This gives a second equilateral triangle, below the side.

Unless the question gives the order of the vertices, give both answers.



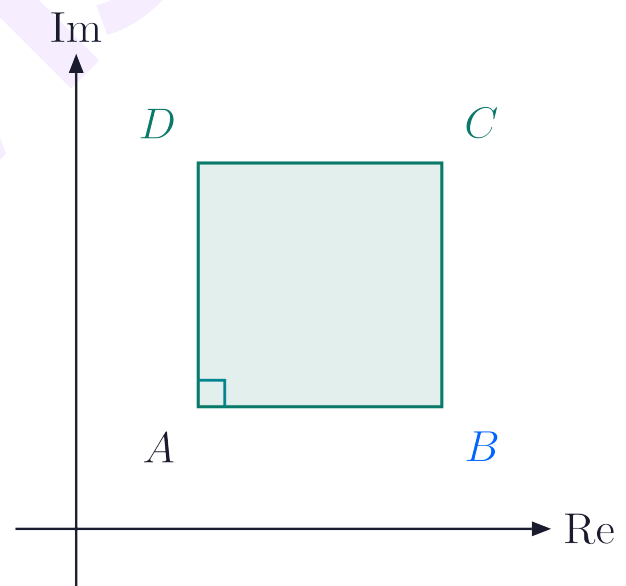
The missing vertices of a square

$ABCD$ is a square with its vertices in anticlockwise order, $A = 1 + i$ and $B = 3 + i$. The side AD is the side AB turned anticlockwise about A by a quarter turn, and C is D moved by the same arrow as A to B :

$$D = A + (B - A)i = (1 + i) + 2i = 1 + 3i,$$

$$C = D + (B - A) = (1 + 3i) + 2 = 3 + 3i.$$

All four sides are 2 and the angle at A is a right angle, so $ABCD$ is a square.



Square $ABCD$ from the side AB

Vertices in anticlockwise order: $D = A + (B - A)i$ and $C = D + (B - A)$. In clockwise order the quarter turn goes the other way, so use $-i$: here that gives $D = 1 - i$ and $C = 3 - i$.

Solved example

A square has its centre at $c = 2 + i$ and one vertex at $v = 4 + 2i$. Find the other three vertices.

The arrow. From the centre to the vertex:
 $v - c = 2 + i$.

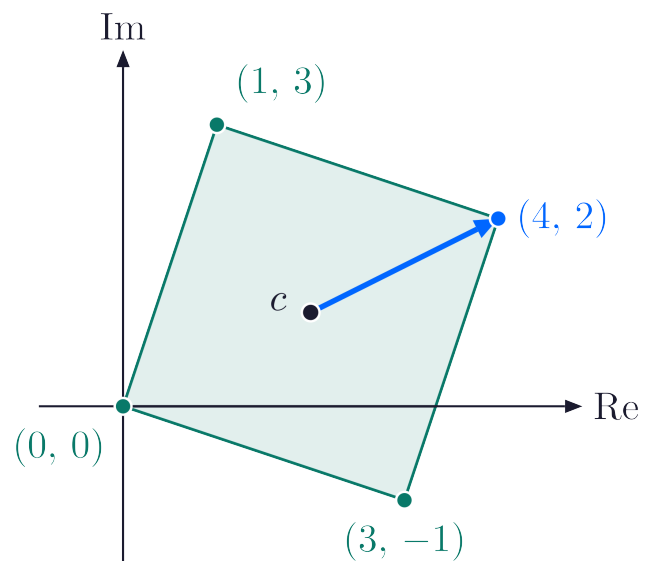
Quarter turns. The vertices of a square are quarter turns of one another about its centre, so multiply the arrow by i , i^2 and i^3 and add c :

$$c + (2 + i)i = (2 + i) + (-1 + 2i) = 1 + 3i$$

$$c + (2 + i)i^2 = (2 + i) + (-2 - i) = 0$$

$$c + (2 + i)i^3 = (2 + i) + (1 - 2i) = 3 - i$$

The other three vertices are $1 + 3i$, 0 and $3 - i$.



Key formulas

Arrow from z_0 to z : $z - z_0$

Turn by θ : multiply by $e^{i\theta}$

About z_0 , anticlockwise: $z' = z_0 + (z - z_0)e^{i\theta}$

Clockwise: $z' = z_0 + (z - z_0)e^{-i\theta}$

Rotation theorem: $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$

Angle: from the denominator side to the numerator side; anticlockwise $+\alpha$, clockwise $-\alpha$

Equilateral on z_1, z_2 : $z_1 + (z_2 - z_1)e^{\pm i\frac{\pi}{3}}$

Square $ABCD$ anticlockwise:
 $D = A + (B - A)i$

Practice set

Easy to hard. Sketch each question first: the picture tells you where the answer should be. The answers start on the next page.

- Q1.** Rotate $4 + 3i$ about $1 + i$ anticlockwise through $\frac{\pi}{2}$.
- Q2.** Rotate $3 + 2i$ about $1 + 2i$ clockwise through $\frac{\pi}{2}$.
- Q3.** Rotate 2 about $1 + i$ through π .
- Q4.** Two vertices of an equilateral triangle are -1 and 1 . Find the third vertex.
- Q5.** Two vertices of an equilateral triangle are $1 + i$ and $1 + 3i$. Find the third vertex.
- Q6.** $ABCD$ is a square with its vertices in anticlockwise order, $A = 0$ and $B = 2 + i$. Find C and D .
- Q7.** Show that 0 , $1 + i$ and $-1 + i$ are the vertices of a right-angled isosceles triangle.
- Q8.** A student rotates $5 + 2i$ about $2 + 2i$ anticlockwise through $\frac{\pi}{2}$ and writes $i(5 + 2i) = -2 + 5i$. Find the mistake and the correct answer.



Watch the animated lesson on YouTube

Rotation Theorem in Complex Numbers

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the Complex Numbers playlist.

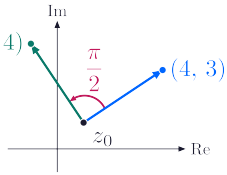
♦ AN INITIATIVE BY AN IIT ALUMNUS ♦

Answers

Q1

$$-1 + 4i$$

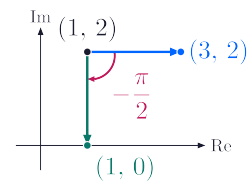
$z - z_0 = 3 + 2i$. Times i : $(-1, 4)$
 $3i + 2i^2 = -2 + 3i$. Add
 z_0 : $(-2 + 3i) + (1 + i) =$
 $-1 + 4i$.



Q2

$$1$$

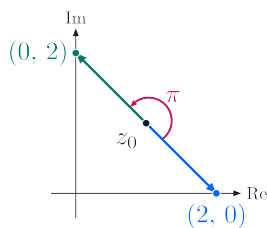
$z - z_0 = 2$. Clockwise,
 so multiply by
 $e^{-i\frac{\pi}{2}} = -i$: $-2i$. Add
 z_0 : $-2i + (1 + 2i) = 1$.



Q3

$$2i$$

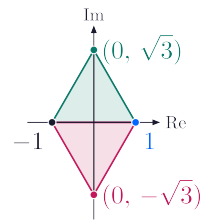
$z - z_0 = 1 - i$. A half
 turn multiplies by
 $e^{i\pi} = -1$: $-1 + i$. Add
 z_0 :
 $(-1 + i) + (1 + i) = 2i$.
 Check:
 $2z_0 - z = 2 + 2i - 2 = 2i$.



Q4

$$\sqrt{3}i \text{ or } -\sqrt{3}i$$

$z_2 - z_1 = 2$, so
 $-1 + 2\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) =$
 $\pm\sqrt{3}i$. No order is
 given, so both.

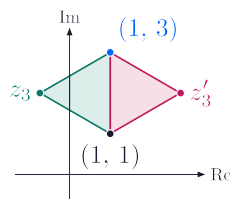


Q5

$$z_3 = (1 - \sqrt{3}) + 2i \text{ or}$$

$$z'_3 = (1 + \sqrt{3}) + 2i$$

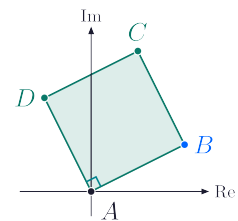
$z_2 - z_1 = 2i$.
 $2i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\sqrt{3} + i$
 and
 $2i\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \sqrt{3} + i$.
 Add $1 + i$ to each.



Q6

$$C = 1 + 3i, D = -1 + 2i$$

$B - A = 2 + i$.
 $D = A + (B - A)i =$
 $2i + i^2 = -1 + 2i$.
 $C = D + (B - A) =$
 $(-1 + 2i) + (2 + i) =$
 $1 + 3i$.



Q7

$$\frac{-1 + i}{1 + i} = i$$

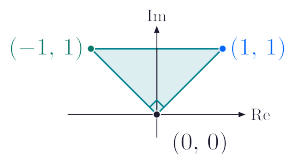
Multiply above and

below by $1 - i$:

$$\frac{(-1+i)(1-i)}{2} = \frac{2i}{2} = i.$$

Modulus 1: the two
 sides from 0 are equal.

Argument $\frac{\pi}{2}$: a right
 angle at 0.



Q8

$$2 + 5i$$

The student turned
 about the origin.

$z - z_0 = 3$; times i : $3i$;
 add z_0 : $2 + 5i$. The
 student's $-2 + 5i$ is
 $|-4 + 3i| = 5$ from z_0 ,
 not 3.

